

Asymptotic expansion of Gaussian integrals of analytic functionals on infinite-dimensional spaces and quantum averages

Andrei Khrennikov
International Center for Mathematical Modeling
in Physics and Cognitive Sciences,
University of Växjö, S-35195, Sweden

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Abstract

We study asymptotic expansions of Gaussian integrals of analytic functionals on infinite-dimensional spaces (Hilbert and nuclear Frechet). We obtain an asymptotic equality coupling the Gaussian integral and the trace of the composition of scaling of the covariation operator of a Gaussian measure and the second (Frechet) derivative of a functional. In this way we couple classical average (given by an infinite-dimensional Gaussian integral) and quantum average (given by the von Neumann trace formula). We can interpret this mathematical construction as a procedure of “dequantization” of quantum mechanics. We represent quantum mechanics as an asymptotic projection of classical statistical mechanics with infinite-dimensional phase-space. This space can be represented as the space of classical fields, so quantum mechanics is represented as a projection of “Prequantum Classical Statistical Field Theory”.

1 Introduction

The problem of reduction of quantum mechanics to classical statistical mechanics has been discussed from the first days of quantum mechanics, see, e.g., [1]–[45]. Now days this problem is known as the problem

of hidden variables or completeness of quantum mechanics, see, e.g., [33] – [37] for recent debates. There is a rather common opinion that quantum mechanics is complete and that it is impossible to introduce “hidden variables” providing more detailed description than quantum mechanics. But in [46] it was demonstrated that in the opposition to this opinion it is possible to represent quantum mechanics as a projection of classical statistical mechanics on *infinite-dimensional space*. In this paper we present this approach (which was called in [46] *Prequantum Classical Statistical Field Theory* – PCSFT) on the mathematical level of rigorousness; in particular, some functional spaces introduced in [46] should be modified to obtain the correct results; moreover, in the basic asymptotic equality coupling classical and quantum averages we obtain an estimate of the rest term, $o(\alpha)$.

In the present paper we also find connection of PCSFT with background Gaussian random field on Hilbert space. Finally, we extend the approach of [46] by considering unbounded operators, see section 6. There are also differences in interpretations of a small parameter of our asymptotic procedure of *dequantization*. In [46] this parameter was identified with the Planck constant \hbar (I was very much stimulated by discussions with people working in SED and stochastic quantum mechanics). In this paper we introduce a new parameter α giving the dispersion of prequantum fluctuations, see [47] for more details on physical interpretation. To simplify considerations, in this paper we consider quantum formalism over the field of real numbers, see [47] for complex theory. To exclude possible misunderstanding, we emphasize from the very beginning that our paper *is not about deformation quantization for systems with the infinite number of degrees of freedom*, see, e.g., [48], [49], but about dequantization of conventional quantum mechanics for systems with a finite number of degrees of freedom by means of analysis on infinite-dimensional space.

Our model is classical statistical mechanics on the phase space $\Omega = H \times H$, where H is the real Hilbert space. Points of this phase-space can be considered as *classical fields* (if we take the Hilbert space $H = L_2(\mathbf{R}^3)$). Our approach can be called *Prequantum Classical Statistical Field Theory* - PCSFT.

Our approach is an asymptotic approach. We introduce a small parameter α – dispersion of “vacuum fluctuations”. In fact we consider a one parameter family of classical statistical models M^α . QM is

obtained as the limit of classical statistical models when $\alpha \rightarrow 0$:

$$\lim_{\alpha \rightarrow 0} M^\alpha = N_{\text{quant}}, \quad (1)$$

where N_{quant} is the Dirac-von Neumann quantum model [2], [4]. As was already remarked, our approach should not be mixed with so called *deformation quantization*. In the formalism of deformation quantization classical mechanics on the phase-space $\Omega_{2n} = \mathbf{R}^{2n}$ is obtained as the $\lim_{h \rightarrow 0}$ of quantum mechanics (the correspondence principle). In the deformation quantization the quantum model is considered as depending on a small parameter h : $N_{\text{quant}} \equiv N_{\text{quant}}^h$, and formally

$$\lim_{h \rightarrow 0} N_{\text{quant}}^h = M_{\text{conv.class.}} \quad (2)$$

where $M_{\text{conv.class.}}$ is the conventional classical model with the phase-space Ω_{2n} .

In our approach the classical \rightarrow quantum correspondence T is based on the Taylor expansion of classical physical variables – functions $f : \Omega \rightarrow \mathbf{R}$. This is a very simple map: function is mapped into its second derivative (which is always a symmetric operator).¹

The space of classical statistical states consists of Gaussian measures on Ω having zero mean value and dispersion $\approx \alpha$. Thus a statistical state ρ (even a so called pure state $\psi \in \Omega, \|\psi\| = 1$) can be interpreted as a Gaussian ensemble of classical fields which are very narrow concentrated near the vacuum field $\psi_{\text{vacuum}}(x) = 0$ for all $x \in \mathbf{R}^3$. Such a ρ has the very small standard quadratic deviation from the field of vacuum ψ_{vacuum} :

$$\int_{L_2(\mathbf{R}^3) \times L_2(\mathbf{R}^3)} \int_{\mathbf{R}^3} [p^2(x) + q^2(x)] dx d\rho(q, p) = \alpha, \quad \alpha \rightarrow 0, \quad (3)$$

where a classical (prequantum) field $\psi(x)$ is a vector field with two components $\psi(x) = (q(x), p(x))$. The field has the dimension of energy per volume (as in the case of electromagnetic field in the Gaussian system of units).² Then a statistical state ρ is an ensemble of fluctuations of vacuum which are small in the energy domain.

¹By the terminology which is used in functional analysis f is called functional – a map from a functional space into real numbers. If we represent Ω as the space of classical fields, $\psi : \mathbf{R}^3 \rightarrow \mathbf{R}$, then $f(\psi)$ is a functional of classical field.

²So we really interpret ψ as a classical field and not as a square root of probability.

The choice of the space of statistical states plays the crucial role in our approach. QM is the image of a very special class of classical statistical states. Therefore we discuss this problem in more detail. Let us use the language of probability theory. Here a statistical state is represented by a Gaussian random variable $\lambda \rightarrow \psi_\lambda$, where λ is a random parameter. We have:

$$E\psi_\lambda = 0, \sigma^2(\psi) = E|\psi_\lambda - \psi_{\text{vacuum}}|^2 = \alpha. \quad (4)$$

We pay attention to the evident fact that small dispersion does not imply that the random variable ψ_λ is small at any point $\lambda \in \Lambda$, where Λ is the space of random parameters. Here smallness is considered with respect to the L_2 -norm. The internal energy of the field ψ_λ ,

$$\mathcal{E}(\psi_\lambda) \equiv \int_{\mathbf{R}^3} |\psi_\lambda(x)|^2 dx = \int_{\mathbf{R}^3} [p_\lambda^2(x) + q_\lambda^2(x)] dx,$$

can be arbitrary large (with nonzero probability). But the probability that $\mathcal{E}(\psi_\lambda)$ is sufficiently large is very small. The easiest way to estimate this probability is to use the (well known in elementary probability theory) Chebyshov inequality:

$$P(\lambda : \mathcal{E}(\psi_\lambda) > C) \leq E\mathcal{E}(\psi_\lambda)/C = \alpha/C \rightarrow 0, \alpha \rightarrow 0, \quad (5)$$

for any constant $C > 0$.

It is especially interesting that in our approach “pure quantum states” are not pure at all! These are also statistical mixtures of small Gaussian fluctuations of the “background field”.

At the moment we are not able to estimate the magnitude α of Gaussian vacuum fluctuations. In the first version of our work [46] we assumed, as it is common in SED [50], [51] as well as in stochastic QM [18], that α has the magnitude of the Planck constant \hbar . However, we could not justify this fundamental assumption on the magnitude of vacuum fluctuations in our approach. It may be that vacuum fluctuations described by PCSFT are essentially smaller than fluctuations considered in SED and stochastic QM. One might even speculate on a connection with *cosmology and string theory*. However, in the present paper we consider the magnitude of vacuum fluctuations just as a small mathematical parameter of the model: $\alpha \rightarrow 0$.

After publication of paper [1], I was informed about the paper of Alexander Bach [52] (see also earlier publications [53], [54]) who also used the Hilbert phase space to construct a classical probabilistic representation of quantum mechanics. In both approaches there was used

the representation of the von Neumann trace formula for quantum averages through integration on the Hilbert phase space. In this sense my approach is a natural development of Bach's approach [52], [53], [54]. However, a new contribution was really nontrivial. Therefore Alexander Bach and I finally came to completely different conclusions on the possibility to reduce quantum mechanics to classical statistical mechanics. We recall the main conclusion of A. Bach [52], p.128 : "Although we give a representation of quantum mechanics in terms of classical probability theory, *the concepts of classical probability theory are not appropriate for quantum theory.*" My main conclusion is completely opposite: quantum mechanics can be represented in a natural way as an approximation of *statistical mechanics of classical fields*.

The main difference between my theory (which was called Prequantum Classical Statistical Field Theory – PCSFT) and Bach's theory is the asymptotic approach to correspondence between the classical and quantum statistical models. In PCSFT there is a small parameter α giving the magnitude of fluctuations in Gaussian ensembles of classical fields.³ We consider not only quadratic functions of fields, but arbitrary smooth functions (as in the classical statistical mechanics). Quantum observables are obtained through expansion of such functions into the Taylor series.

This viewpoint to quantum mechanics – as the second order approximation of classical statistical mechanics on Hilbert phase space – gives the possibility to solve a problem that was crucial for Bach's model (and as we see from his article [52] that problem was the main reason for rather pessimistic Bach's conclusion which was mentioned above). This is the problem of correspondence between functions of classical physical variables and functions of operators. If f is a classical variable (in our approach an arbitrary smooth function on the Hilbert phase space and in Bach's approach a quadratic form) and $T(f)$ is the corresponding quantum observable (a self-adjoint operator), then, for example,

$$T(f^2) \neq T(f)^2. \quad (6)$$

This is not a purely mathematical problem. As was pointed out by Alexander Bach, this is the root of difference in the definition of dispersion free states in the quantum model and in a prequantum classical statistical model with the Hilbert phase space.

³We remark that we consider fluctuations in ensembles of classical fields, but not fluctuations of a single field on physical space \mathbf{R}^3 .

It was totally impossible to solve this problem in Bach's framework. His prequantum model was an exact one. Therefore the violation of the equality $T(f^2) = T(f)^2$ was considered as the evidence of inadequacy of this model to quantum mechanics. Our model, PCSFT, is not an exact model. This is an asymptotic model or better to say an a prequantum model which is approximated by the quantum model.

However, *the violation of, e.g., the equality $T(f^2) = T(f)^2$ in some approximation scheme was never considered in approximation theory as an evidence of inadequacy of this scheme.* For example, let us consider the approximation of smooth functions $f : \mathbf{R}^n \rightarrow \mathbf{R}$ by their Taylor polynomials of the order m . This approximation scheme induces the map

$$T : C^\infty \rightarrow \mathcal{P}_m, \quad (7)$$

where C^∞ and \mathcal{P}_m are, respectively, spaces of smooth functions and polynomials of the degree m . Then it is evident that (as in our prequantum model), e.g., the equality $T(f^2) = T(f)^2$ can be violated. But nobody would conclude that physics described by polynomials of the degree m (e.g., $m = 2$) differs crucially from physics described by smooth functions.

We finish our comparative analysis with Bach's model by emphasizing that PCSFT provides the natural interpretation of hidden variables: these are classical fields. But in [52] there was still pointed out that "... the fact that elements of Hilbert space have no empirical meaning indicates that the theory still remains open to interpretations."

We also consider generalizations of quantum formalism based on expansions of functionals of classical fields into the Taylor series up to terms of degree n , see section 7; for $n = 2$ we obtain the conventional quantum mechanics.

2 Infinite-dimensional analysis

Gaussian stochastic analysis on infinite-dimensional spaces is a well established mathematical formalism, see, e.g., Skorohod [55] for introduction, see [56], [57], [58] for more detail (especially for applications in sections 5.2 and 6). We also pay attention that Gaussian analysis on infinite-dimensional spaces was used a lot in Euclidean quantum field theory, see, e.g., [59], [60].

Let H be a real Hilbert space and let $A : H \rightarrow H$ be a continuous self-adjoint linear operator. The basic mathematical formula which will be used in this paper is the formula for a Gaussian integral of a quadratic form $f(\psi) \equiv f_A(\psi) = (A\psi, \psi)$.

Let $d\rho(\psi)$ be a σ -additive Gaussian measure on the σ -field F of Borel subsets of H , see [56]–[59]. This measure is determined by its covariation operator $B : H \rightarrow H$ and mean value $m \equiv m_\rho \in H$. For example, B and m determines the Fourier transform of $\rho : \tilde{\rho}(y) = \int_H e^{i(y, \psi)} d\rho(\psi) = e^{\frac{1}{2}(By, y) + i(m, y)}$, $y \in H$. In what follows we restrict our considerations to Gaussian measures with zero mean value $m = 0$, where $(m, y) = \int_H (y, \psi) d\rho(\psi) = 0$ for any $y \in H$. Sometimes there will be used the symbol ρ_B to denote the Gaussian measure with the covariation operator B and $m = 0$. We recall that the covariation operator $B \equiv \text{cov } \rho$ is defined by $(By_1, y_2) = \int (y_1, \psi)(y_2, \psi) d\rho(\psi)$, $y_1, y_2 \in H$, and has the following properties: a). $B \geq 0$, i.e., $(By, y) \geq 0$, $y \in H$; b). B is a self-adjoint operator, $B \in \mathcal{L}_s(H)$; c). B is a trace-class operator and $\text{Tr } B = \int_H \|\psi\|^2 d\rho(\psi)$. This is *dispersion* $\sigma^2(\rho)$ of the probability ρ . Thus $\sigma^2(\rho) = \text{Tr } B$.

We pay attention that the list of properties of the covariation operator of a Gaussian measure differs from the list of properties of a von Neumann density operator [4] only by one condition: $\text{Tr } D = 1$, for a density operator D .

We can easily find the Gaussian integral of the quadratic form $f_A(\psi)$:

$$\int_H f_A(\psi) d\rho(\psi) = \text{Tr } BA \quad (8)$$

The differential calculus for maps $f : H \rightarrow \mathbf{R}$ does not differ so much from the differential calculus in the finite dimensional case, $f : \mathbf{R}^n \rightarrow \mathbf{R}$. Instead of the norm on \mathbf{R}^n , one should use the norm on H . We consider so called Frechet differentiability. Here a function f is differentiable if it can be represented as $f(\psi_0 + \Delta\psi) = f(\psi_0) + f'(\psi_0)(\Delta\psi) + o(\Delta\psi)$, where $\lim_{\|\Delta\psi\| \rightarrow 0} \frac{\|o(\Delta\psi)\|}{\|\Delta\psi\|} = 0$. Here at each point ψ the derivative $f'(\psi)$ is a continuous linear functional on H ; so it can be identified with the element $f'(\psi) \in H$. Then we can define the second derivative as the derivative of the map $\psi \rightarrow f'(\psi)$ and so on. A map f is differentiable n -times iff:

$$\begin{aligned} f(\psi_0 + \Delta\psi) &= f(\psi_0) + f'(\psi_0)(\Delta\psi) + \frac{1}{2}f''(\psi_0)(\Delta\psi, \Delta\psi) + \dots \\ &+ \frac{1}{n!}f^{(n)}(\psi_0)(\Delta\psi, \dots, \Delta\psi) + o_n(\Delta\psi), \end{aligned}$$

where $f^{(n)}(\psi_0)$ is a symmetric continuous n -linear form on H and $\lim_{\|\Delta\psi\| \rightarrow 0} \frac{\|o_n(\Delta\psi)\|}{\|\Delta\psi\|^n} = 0$. For us it is important that $f''(\psi_0)$ can be represented by a symmetric operator $f''(\psi_0)(u, v) = (f''(\psi_0)u, v)$, $u, v \in H$ (this fact is well known in the finite dimensional case: the matrix representing the second derivative of any two times differentiable function $f : \mathbf{R}^n \rightarrow \mathbf{R}$ is symmetric). We remark that in this case $f(\psi) = f(0) + f'(0)(\psi) + \frac{1}{2}f''(0)(\psi, \psi) + \dots + \frac{1}{n!}f^{(n)}(0)(\psi, \dots, \psi) + o_n(\psi)$.

For a real Hilbert space H , denote by the symbol $H^{\mathbf{C}}$ its complexification: $H^{\mathbf{C}} = H \oplus iH$. We recall that a function $f : H^{\mathbf{C}} \rightarrow \mathbf{C}$ is analytic if it can be expanded into the Taylor series:

$$f(\psi) = f(0) + f'(0)(\psi) + \frac{1}{2}f''(0)(\psi, \psi) + \dots + \frac{1}{n!}f^{(n)}(0)(\psi, \dots, \psi) + \dots \quad (9)$$

which converges uniformly on any ball of $H^{\mathbf{C}}$.

3 Dequantization

3.1 Classical and quantum statistical models

We define “*classical statistical models*” in the following way, see [46] for more detail (and even philosophic considerations): a) physical states ω are represented by points of some set Ω (state space); b) physical variables are represented by functions $f : \Omega \rightarrow \mathbf{R}$ belonging to some functional space $V(\Omega)$; c) statistical states are represented by probability measures on Ω belonging to some class $S(\Omega)$; d) the average of a physical variable (which is represented by a function $f \in V(\Omega)$) with respect to a statistical state (which is represented by a probability measure $\rho \in S(\Omega)$) is given by

$$\langle f \rangle_{\rho} \equiv \int_{\Omega} f(\psi) d\rho(\psi). \quad (10)$$

A *classical statistical model* is a pair $M = (S, V)$. We recall that classical statistical mechanics on the phase space $\Omega_{2n} = \mathbf{R}^n \times \mathbf{R}^n$ gives an example of a classical statistical model. But we shall not be interested in this example in our further considerations. We shall develop a classical statistical model with *an infinite-dimensional phase-space*.

In real Hilbert space H a quantum statistical model is described in the following way (see Dirac-von Neumann [2], [4] for the conventional

complex model): a) physical observables are represented by operators $A : H \rightarrow H$ belonging to the class of continuous self-adjoint operators $\mathcal{L}_s \equiv \mathcal{L}_s(H)$; b) statistical states are represented by von Neumann density operators, see [4] (the class of such operators is denoted by $\mathcal{D} \equiv \mathcal{D}(H)$); d) the average of a physical observable (which is represented by the operator $A \in \mathcal{L}_s(H)$) with respect to a statistical state (which is represented by the density operator $D \in \mathcal{D}(H)$) is given by von Neumann's formula [4]:

$$\langle A \rangle_D \equiv \text{Tr } DA \quad (11)$$

The *quantum statistical model* is the pair $N_{\text{quant}} = (\mathcal{D}, \mathcal{L}_s)$.

3.2 Asymptotic equality of classical and quantum averages and amplification of classical variables

We are looking for a classical statistical model $M = (S, V)$ which will give “dequantization” of the quantum model $N_{\text{quant}} = (\mathcal{D}, \mathcal{L}_s)$. Here the meaning of “dequantization” should be specified. In fact, all “NO-GO” theorems (e.g., von Neumann, Kochen-Specker, Bell,...) can be interpreted as theorems about impossibility of various dequantization procedures. Therefore we should define the procedure of dequantization in such a way that there will be no contradiction with known “NO-GO” theorems, but our dequantization procedure still will be natural from the physical viewpoint. We define (asymptotic) dequantization as a family $M^\alpha = (S^\alpha, V)$ of classical statistical models depending on small parameter α . There should exist maps $T : S^\alpha \rightarrow \mathcal{D}$ and $T : V \rightarrow \mathcal{L}_s$ such that: a) both maps are *surjections* (so all quantum objects are covered by classical); b) the map $T : V \rightarrow \mathcal{L}_s$ is linear; c) the map $T : S \rightarrow \mathcal{D}$ is injection (there is one-to one correspondence between classical and quantum statistical states); d) classical and quantum averages are coupled through the following asymptotic equality:

$$\langle f \rangle_\rho = \alpha \langle T(f) \rangle_{T(\rho)} + o(\alpha), \quad \alpha \rightarrow 0 \quad (12)$$

(here $\langle T(f) \rangle_{T(\rho)}$ is the quantum average). In mathematical models this equality has the form:

$$\int_{\Omega} f(\psi) d\rho(\psi) = \alpha \text{Tr } DA + o(\alpha), \quad A = T(f), D = T(\rho). \quad (13)$$

This equality can be interpreted in the following way. Let $f(\psi)$ be a classical physical variable (describing properties of microsystems - classical fields having very small magnitude α). We define its *amplification* by:

$$f_\alpha(\psi) = \frac{1}{\alpha} f(\psi) \quad (14)$$

(so any micro effect is amplified in $\frac{1}{\alpha}$ -times). Then we have:

$$\langle f_\alpha \rangle_\rho = \langle T(f) \rangle_{T(\rho)} + o(1), \quad \alpha \rightarrow 0, \quad (15)$$

or

$$\int_{\Omega} f_\alpha(\psi) d\rho(\psi) = \text{Tr } DA + o(1), \quad A = T(f), D = T(\rho). \quad (16)$$

Thus: *Quantum average \approx Classical average of the $\frac{1}{\alpha}$ -amplification.*
Hence: *QM is a mathematical formalism describing a statistical approximation of amplification of micro effects.*

We see that for physical variables/quantum observables and classical and quantum statistical states the dequantization maps have different features. The map $T : V \rightarrow \mathcal{L}_s$ is not injective. Different classical physical variables f_1 and f_2 can be mapped into one quantum observable A . This is not surprising. Such a viewpoint on the relation between classical variables and quantum observables was already presented by J. Bell, see [12]. In principle, experimenter could not distinguish classical (“ontic”) variables by his measurement devices. In contrast, the map $T : S^\alpha \rightarrow \mathcal{D}$ is injection. Here we suppose that quantum statistical states represent uniquely (“ontic”) classical statistical states.

The crucial difference with dequantizations considered in known “NO-GO” theorems is that in our case classical and quantum averages are equal only asymptotically and that a classical variable f and the corresponding quantum observable $A = T(f)$ can have different ranges of values.

3.3 Asymptotic Gaussian analysis

Let us consider a classical statistical model in that the state space $\Omega = H$ (in physical applications $H = L_2(\mathbf{R}^3)$ is the space of classical fields on \mathbf{R}^3) and the space of statistical states consists of Gaussian measures with zero mean value and dispersion

$$\sigma^2(\rho) = \int_{\Omega} \|\psi\|^2 d\rho(\psi) = \alpha, \quad (17)$$

where $\alpha > 0$ is a small real parameter. Denote such a class of Gaussian measures by the symbol $S_G^\alpha(\Omega)$. For $\rho \in S_G^\alpha(\Omega)$, we have $\text{Tr cov } \rho = \alpha$. We remark that any linear transformation (in particular, scaling) preserves the class of Gaussian measures. Let us make the change of variables (scaling):

$$\psi \rightarrow \frac{\psi}{\sqrt{\alpha}}. \quad (18)$$

(we emphasize that this is a scaling not in the physical space \mathbf{R}^3 , but in the space of fields on it). To find the covariation operator D of the image ρ_D of the Gaussian measure ρ_B , we compute its Fourier transform: $\tilde{\rho}_D(\xi) = \int_\Omega e^{i(\xi, y)} d\rho_D(y) = \int_\Omega e^{i(\xi, \frac{\psi}{\sqrt{\alpha}})} d\rho_B(\psi) = e^{-\frac{1}{2\alpha}(B\xi, \xi)}$. Thus

$$D = \frac{B}{\alpha} = \frac{\text{cov } \rho}{\alpha}. \quad (19)$$

We shall use this formula later. We remark that by definition:

$$\langle f \rangle_{\rho_B} = \int_\Omega f(\psi) d\rho_B(\psi) = \int_\Omega f(\sqrt{\alpha}\psi) d\rho_D(\psi).$$

To make our further considerations mathematically rigorous, we should attract the theory of analytic functions $f : \Omega^{\mathbf{C}} \rightarrow \mathbf{C}$. Here $\Omega^{\mathbf{C}} = \Omega \oplus i\Omega$ is the complexification of the real Hilbert space Ω .

Let $b_n : \Omega^{\mathbf{C}} \times \dots \times \Omega^{\mathbf{C}} \rightarrow \mathbf{C}$ be a continuous n -linear symmetric form. We define its norm by $\|b_n\| = \sup_{\|\psi\| \leq 1} |b_n(\psi, \dots, \psi)|$. Thus

$$|b_n(\psi, \dots, \psi)| \leq \|b_n\| \|\psi\|^n \quad (20)$$

Let us consider the space analytic functions of the *exponential growth*:

$$|f(\psi)| \leq ae^{b\|\psi\|}, \psi \in \Omega^{\mathbf{C}}, \quad (21)$$

see, e.g., [60]. Here constants depend on f : $a = a_f, b = b_f$.

Lemma 3.1 *The space of analytic functions of the exponential growth coincides with the space of analytic functions such that:*

$$\|f^{(n)}(0)\| \leq c r^n, \quad n = 0, 1, 2, \dots \quad (22)$$

Here constants $c = c_f$ and $r = r_f$ depend on the function f .

Proof. A). Let f have the exponential growth. For any $\psi \in \Omega^{\mathbf{C}}$, we consider the function of the complex variable $z \in \mathbf{C}$: $g_\psi(z) = f(z\psi)$. By the Cauchy integral formula for $g_\psi(z)$ we have: $g_\psi^{(n)}(0) =$

$\frac{n!}{2\pi i} \int_{|z|=R} g_\psi(z) z^{-(n+1)} dz$, where at the moment $R > 0$ is a free parameter. Thus: $|g_\psi^{(n)}(0)| \leq n! R^{-n} \sup_{0 \leq \theta \leq 2\pi} |f(Re^{i\theta}\psi)| \leq a_f n! R^{-n} e^{b_f R \|\psi\|}$. By choosing $R = n$ and observing that $g_\psi^{(n)}(0) = f^{(n)}(0)(\psi, \dots, \psi)$ we obtain:

$$\|f^{(n)}(0)\| \leq a'_f e^{-n} n^{1/2} e^{b_f n}.$$

Thus the derivatives of f satisfy the inequalities (22) with $r_f = e^{b_f}$.

B). Let now derivatives of f satisfy the inequalities (22). Then by the inequalities (20) we have $|f(\psi)| \leq \sum_{n=0}^{\infty} \|f^{(n)}(0)\| \|\psi\|^n / n! \leq c_f \sum_{n=0}^{\infty} (r_f \|\psi\|)^n / n! \leq c_f e^{r_f \|\psi\|}$. Thus f has the exponential growth with $b_f = r_f$.

We denote by the symbol $\mathcal{V}(\Omega)$ the following space of functions $f : \Omega \rightarrow \mathbf{R}$. Each $f \in \mathcal{V}(\Omega)$ takes the value zero at the point $\psi = 0$ and it can be extended to the analytic function $f : \Omega^{\mathbf{C}} \rightarrow \mathbf{C}$ having the exponential growth.

Example 3.1. In particular, any polynomial on Ω belongs to the space $\mathcal{V}(\Omega)$. For example, let A_1, \dots, A_N be continuous linear operators. Then function $f(\psi) = \sum_{n=1}^N (A_n \psi, \psi)^n$ belongs to the space $\mathcal{V}(\Omega)$.

Any function $f \in \mathcal{V}(\Omega)$ is integrable with respect to any Gaussian measure on Ω , see [55]. Let us consider the family of the classical statistical models

$$M^\alpha = (S_G^\alpha(\Omega), \mathcal{V}(\Omega)).$$

Let a variable $f \in \mathcal{V}(\Omega)$ and let a statistical state $\rho_B \in S_G^\alpha(\Omega)$. Our further aim is to find an asymptotic expansion of the (classical) average $\langle f \rangle_{\rho_B} = \int_{\Omega} f(\psi) d\rho_B(\psi)$ with respect to the small parameter α .

Lemma 3.2. *Let $f \in \mathcal{V}(\Omega)$ and let $\rho \in S_G^\alpha(\Omega)$. Then the following asymptotic equality holds:*

$$\langle f \rangle_{\rho} = \frac{\alpha}{2} \text{Tr } D f''(0) + o(\alpha), \quad \alpha \rightarrow 0, \quad (23)$$

where the operator D is given by (19). Here

$$o(\alpha) = \alpha^2 R(\alpha, f, \rho), \quad (24)$$

where $|R(\alpha, f, \rho)| \leq c_f \int_{\Omega} e^{r_f \|\psi\|} d\rho_D(\psi)$.

Proof. In the Gaussian integral $\int_{\Omega} f(\psi) d\rho(\psi)$ we make the scaling (18):

$$\langle f \rangle_{\rho} = \int_{\Omega} f(\sqrt{\alpha}\psi) d\rho_D(\psi) = \frac{\alpha}{2} \int_{\Omega} (f''(0)\psi, \psi) d\rho_D(\psi) + \alpha^2 R(\alpha, f, \rho), \quad (25)$$

where

$$R(\alpha, f, \rho) = \int_{\Omega} g(\alpha, f; \psi) d\rho_D(\psi), g(\alpha, f; \psi) = \sum_{n=4}^{\infty} \frac{\alpha^{n/2-2}}{n!} f^{(n)}(0)(\psi, \dots, \psi).$$

We pay attention that

$$\int_{\Omega} (f'(0), \psi) d\rho_D(\psi) = 0, \quad \int_{\Omega} f'''(0)(\psi, \psi, \psi) d\rho_D(\psi) = 0,$$

because the mean value of ρ (and, hence, of ρ_D) is equal to zero. Since $\rho \in S_G^{\alpha}(\Omega)$, we have $\text{Tr } D = 1$.

The change of variables in (25) can be considered as scaling of the magnitude of statistical (Gaussian) fluctuations. Negligibly small random fluctuations $\sigma(\rho) = \sqrt{\alpha}$ (where α is a small parameter) are considered in the new scale as standard normal fluctuations. If we use the language of probability theory and consider a Gaussian random variables $\xi(\lambda)$, then the transformation (18) is nothing else than the standard normalization of this random variable (which is used, for example, in the central limit theorem): $\eta(\lambda) = \frac{\xi(\lambda) - E\xi}{\sqrt{E(\xi(\lambda) - E\xi)^2}}$ (in our case $E\xi = 0$).

We now estimate the rest term $R(\alpha, f, \rho)$. By using the inequality (22) we have for $\alpha \leq 1$:

$$|g(\alpha, f; \psi)| = \sum_{n=4}^{\infty} \frac{\|f^{(n)}(0)\| \|\psi\|^n}{n!} \leq c_f \sum_{n=4}^{\infty} \frac{r_f^n \|\psi\|^n}{n!} = C_f e^{r_f \|\psi\|}.$$

Thus: $|R(\alpha, f, \rho)| \leq c_f \int_{\Omega} e^{r_f \|\psi\|} d\rho_D(\psi)$. We obtain:

$$\langle f \rangle_{\rho} = \frac{\alpha}{2} \int_{\Omega} (f''(0)\psi, \psi) d\rho_D(\psi) + o(\alpha), \quad \alpha \rightarrow 0. \quad (26)$$

By using the equality (8) we finally come the asymptotic equality (23).

We see that the classical average (computed in the model $M^{\alpha} = (S_G^{\alpha}(\Omega), \mathcal{V}(\Omega))$ by using the measure-theoretic approach) is coupled through (23) to the quantum average (computed in the model $N_{\text{quant}} = (\mathcal{D}(\Omega), \mathcal{L}_s(\Omega))$ by the von Neumann trace-formula).

The equality (23) can be used as the motivation for defining the following classical \rightarrow quantum map T from the classical statistical model $M^{\alpha} = (S_G^{\alpha}, \mathcal{V})$ onto the quantum statistical model $N_{\text{quant}} = (\mathcal{D}, \mathcal{L}_s)$:

$$T : S_G^{\alpha}(\Omega) \rightarrow \mathcal{D}(\Omega), \quad D = T(\rho) = \frac{\text{cov } \rho}{\alpha} \quad (27)$$

(the Gaussian measure ρ is represented by the density matrix D which is equal to the covariation operator of this measure normalized by α);

$$T : \mathcal{V}(\Omega) \rightarrow \mathcal{L}_s(\Omega), \quad A_{\text{quant}} = T(f) = \frac{1}{2}f''(0). \quad (28)$$

Our previous considerations can be presented as

Theorem 3.1. *The one parametric family of classical statistical models $M^\alpha = (S_G^\alpha, \mathcal{V})$ provides dequantization of the quantum model $N_{\text{quant}} = (\mathcal{D}, \mathcal{L}_s)$ through the pair of maps (27) and (28). The classical and quantum averages are coupled by the asymptotic equality (23).*

4 Gaussian underground for pure states

In quantum mechanics a pure quantum state is given by a normalized vector $\Psi \in H : \|\Psi\| = 1$. In our model such a state is not pure at all (in the sense that such a vector Ψ does not provide a description of an individual system). Such a normalized vector Ψ is the label of a Gaussian statistical mixture. The corresponding quantum statistical state is represented by the density operator: $D_\Psi = \Psi \otimes \Psi$. In particular, the von Neumann's trace-formula for expectation has the form: $\text{Tr } D_\Psi A = (A\Psi, \Psi)$. Let us consider the correspondence map T for statistical states for the classical statistical model $M^\alpha = (S_G^\alpha, \mathcal{V})$, see (27). A pure quantum state Ψ (i.e., the state with the density operator D_Ψ) is the image of the Gaussian statistical mixture ρ_Ψ of states $\psi \in H$. We use the capital Ψ to denote a quantum pure state. This is just the special system of labeling of the Gaussian measure ρ_Ψ by the normalized vector Ψ of Hilbert space. Points of the sample space on that this measure is defined we denote by the low ψ . The measure ρ_Ψ has the covariation operator $B_\Psi = \alpha D_\Psi$. This means that the measure ρ_Ψ is concentrated on the one-dimensional subspace $H_\Psi = \{x \in H : x = s\Psi, s \in \mathbf{R}\}$. This is one-dimensional Gaussian distribution.

5 Pure states as one-dimensional projections of spatial white-noise

In section 4 we showed that so called pure states of quantum mechanics have the natural classical statistical interpretation as Gaussian measures concentrated on one-dimensional subspaces of the Hilbert space

H . On the other hand, it is well known that *any Gaussian measure on H is determined by its one-dimensional projections*. To determine a Gaussian random variable $\xi(\omega) \in H$, it is sufficient to determine all its one-dimensional projections: $\xi_\Psi(\omega) = (\Psi, \xi(\omega))$, $\Psi \in H$. The co-variation operator B of ξ (having the zero mean value) is defined by $(B\Psi, \Psi) = E\xi_\Psi^2$. We are interested in the following problem:

Is it possible to construct a Gaussian distribution on H such that its one-dimensional projections will give us all pure quantum states, $\Psi \in H$, $\|\Psi\| = 1$?

We recall that in our approach a pure quantum state Ψ is just the label for a Gaussian random variable ξ_Ψ such that $E\xi_\Psi^2 = \alpha\|\Psi\|^2 = \alpha$. Thus the answer to our question is positive and pure quantum states can be considered as one-dimensional projections of the $\sqrt{\alpha}$ -scaling of the standard Gaussian distribution on H . The standard Gaussian distribution μ on H (so the average of μ is equal to zero and $\text{cov } \mu = I$, where I is the unit operator) is nothing else than the *white noise* on \mathbf{R}^3 (if one chooses $H = L_2(\mathbf{R}^3)$), see [56]–[59] for details. Thus pure quantum states are simply one-dimensional projections of the *spatial white noise*. It is well known, see, e.g., [56]–[59], that the μ is *not σ -additive on the σ -field of Borel subsets of H* .

To escape mathematical difficulties and concentrate on the dequantization of quantum mechanics, we start with consideration of the finite-dimensional case.

5.1 The finite-dimensional case

We consider the family of Gaussian random variables $\xi_\Psi, \Psi \in \mathbf{R}^n$, $E\xi_\Psi = 0, E\xi_\Psi^2 = \alpha\|\Psi\|^2$. This family can be realized as $\xi_\Psi(\omega) = (\Psi, \xi(\omega))$ where $\xi(\omega) = \sqrt{\alpha}\eta(\omega)$ and $\eta(\omega) \in \mathbf{R}^n$ is standard Gaussian random variable (so $E\eta = 0, \text{cov } \eta = I$). For any $\Psi \in \mathbf{R}^n$, we define the projection P_Ψ to this vector: $P_\Psi(k) = (\Psi, k)\Psi$.

Denote by the symbol $\mathcal{V}(\mathbf{R}^n)$ the class of functions $f : \mathbf{R}^n \rightarrow \mathbf{R}$ such that $f(0) = 0$ and f can be continued analytically onto \mathbf{C}^n and this continuation $f(z)$ has the exponential growth.

Proposition 5.1 *Let $f \in \mathcal{V}(\mathbf{R}^n)$. Then we have:*

$$Ef(P_\Psi\xi(\omega)) = \frac{\alpha\|\Psi\|^2}{2}(f''(0)\Psi, \Psi) + o(\alpha), \alpha \rightarrow 0. \quad (29)$$

Proof. By using the Taylor expansion of f we obtain:

$$Ef(P_\Psi \xi(\omega)) = \frac{1}{2}E(\Psi, \xi(\omega))^2(f''(0)\Psi, \Psi) + o(\alpha), \alpha \rightarrow 0.$$

By setting into this asymptotic equality the dispersion of the random variable $P_\Psi \xi(\omega)$ we obtain (29).

If $\|\Psi\| = 1$ (a pure quantum state), then we get:

$$Ef(P_\Psi \xi(\omega)) = \frac{\alpha}{2}(f''(0)\Psi, \Psi) + o(\alpha), \alpha \rightarrow 0. \quad (30)$$

Here $A = f''(0)$ is a symmetric linear operator. We “quantize” the classical variable $f(x), x \in \mathbf{R}^n$, by mapping it to the operator $A = \frac{1}{2}f''(0)$, see Theorem 3.1. The Gaussian random variable $\xi_\Psi, \|\Psi\| = 1$ is “quantize” by mapping it into the pure quantum state Ψ .

Theorem 5.1. *There exists a Kolmogorov probability space such that all pure quantum states can be represented by Gaussian random variables on this space. The correspondence $\Psi \rightarrow \xi_\Psi(\omega)$ is linear:*

$$\lambda_1 \Psi_1 + \lambda_2 \Psi_2 \rightarrow \lambda_1 \xi_{\Psi_1}(\omega) + \lambda_2 \xi_{\Psi_2}(\omega), \quad (31)$$

where $\lambda_1, \lambda_2 \in \mathbf{R}$.

Proof. We choose $\Omega = \mathbf{R}^n$ as the space of elementary events, the σ -field of Borel subsets is the space of events and the standard Gaussian measure μ as the probability measure. Then for $\Psi \rightarrow \xi_\Psi(\omega) = \sqrt{\alpha}(\Psi, \omega), \omega \in \mathbf{R}^n$, we have: $\lambda_1 \xi_{\Psi_1}(\omega) + \lambda_2 \xi_{\Psi_2}(\omega) = \lambda_1(\Psi_1, \omega) + \lambda_2(\Psi_2, \omega) = (\lambda_1 \Psi_1 + \lambda_2 \Psi_2, \omega) = \xi_{\lambda_1 \Psi_1 + \lambda_2 \Psi_2}(\omega)$.

This theorem is rather surprising from the common viewpoint (by that essentially nonclassical probabilistic features of quantum states are consequences of the *non-Kolmogorovian structure* of the quantum probabilistic model).

We pay attention that physical variables $\xi_\Psi(\omega) = P_\Psi \xi(\omega), \Psi \in \mathbf{R}^n$, (one-dimensional projections of the scaling $\xi(\omega)$ of the standard Gaussian random variable $\eta(\omega) \in \mathbf{R}^n$) *cannot be mapped onto nontrivial quantum observables*. Prequantum classical physical variables $\xi_\Psi(\omega) = (\Psi, \omega)$ are linear functionals of ω . Therefore $T(\xi_\Psi) = \xi_\Psi''(0) = 0$. Nevertheless, quantum mechanics contains images of ξ_Ψ given by quantum states Ψ , but only for Ψ with $\|\Psi\| = 1$!

We call $\xi(\omega)$ a *background random field*. All pure states could be extracted from the the background random field by projecting it to one dimensional subspaces. PCSFT explains the origin of the scalar

product on the set of pure quantum states. We consider the $1/\alpha$ -amplification of the covariation of two Gaussian (prequantum) random variables $\xi_{\Psi_1}(\omega)$ and $\xi_{\Psi_2}(\omega)$. We have:

$$\frac{1}{\alpha} E \xi_{\Psi_1}(\omega) \xi_{\Psi_2}(\omega) = (\Psi_1, \Psi_2). \quad (32)$$

Conclusion. *The Hilbert space structure of quantum mechanics is induced by the (prequantum) Gaussian random field (the background field $\xi(\omega)$) through the $\alpha \rightarrow 0$ asymptotic.*

At the moment we proved this only in the finite-dimensional case. In section 5.2 we shall do this in the infinite-dimensional case. Finally, we pay attention to the fact that, for quadratic physical variables $f(x) = \frac{1}{2}(Ax, x)$, where $A : H \rightarrow H$ is a symmetric operator, the asymptotic equality (30) is reduced to the precise equality of averages. By considering directly the standard Gaussian random variable $\eta(\omega)$ (instead of the background random field $\xi(\omega) = \sqrt{\alpha}\eta(\omega)$) we come to the following classical probabilistic representation of the quantum average: $E f(P_\Psi \eta(\omega)) = (A\Psi, \Psi) \equiv \langle A \rangle_\Psi$.

5.2 Prequantum white noise field

To repeat consideration of section 5.1 for the infinite-dimensional case, we consider measures on the so called rigged Hilbert spaces. We apply some rather abstract mathematical constructions. However, finally we shall consider a simple concrete example which will be then used as the basis of our prequantum classical statistical model.

Let Ω be a nuclear Frechet⁴ topological linear space and Ω' its dual space. Suppose that Ω is densely and continuously embedded into a Hilbert space H , so $\Omega \subset H$. Thus the dual space H' is densely embedded into Ω' . By identifying H and H' we obtain the rigged Hilbert space:

$$\Omega \subset H \subset \Omega' \quad (33)$$

In our final application we shall set $\Omega = \mathcal{S}(\mathbf{R}^3)$. This is the space of Schwartz test functions on \mathbf{R}^3 . Here $\Omega' = \mathcal{S}'(\mathbf{R}^3)$ is the space of Schwartz distributions. In this case we choose $H = L_2(\mathbf{R}^3)$ and we shall consider the rigged Hilbert space:

$$\mathcal{S}(\mathbf{R}^3) \subset L_2(\mathbf{R}^3) \subset \mathcal{S}'(\mathbf{R}^3) \quad (34)$$

⁴so complete metrizable and locally convex

Readers who are not so much interested in general theory of topological linear spaces can consider this rigged Hilbert space throughout this section.

A Gaussian measure ρ on Ω' is determined by its characteristic functional (Fourier transform) $\tilde{\rho}$ which is defined on Ω : $\tilde{\rho}(\Psi) = e^{-\frac{1}{2}b(\Psi, \Psi)}$, where $b : \Omega \times \Omega \rightarrow \mathbf{R}$ is a continuous positively defined quadratic form. By the well known theorem of Minlos-Sazonov, see e.g., [1], ρ is σ -additive on Ω' and its covariation functional is equal to b . Here $b(\Psi_1, \Psi_2) = \int_{\Omega'} (\phi, \Psi_1)(\phi, \Psi_2) d\rho(\phi)$, where $\Psi_1, \Psi_2 \in \Omega$. This functional defines the covariation operator $B : \Omega \rightarrow \Omega'$ by $(B\Psi_1, \Psi_2) = b(\Psi_1, \Psi_2)$. This operator is self-adjoint in the following sense. The dual operator $B' : \Omega'' \rightarrow \Omega'$. But, since the topological linear space Ω is a nuclear Frechet space, it is *reflexive*. Hence, $\Omega'' = \Omega$. Thus the operator $B' : \Omega \rightarrow \Omega'$. Thus it is meaningful to speak about self-adjoint operators in this framework (by extending the ordinary theory of self-adjoint operators in Hilbert space). We also pay attention to the fact that the covariation operator B is positively defined.

Let us consider the standard Gaussian distribution μ on H that is defined by its covariation functional:

$$b(\Psi_1, \Psi_2) = (\Psi_1, \Psi_2).$$

The corresponding covariation operator $B = I : \Omega \rightarrow \Omega'$ is the canonical embedding operator. Since the embedding $\Omega \subset H$ is continuous, $b : \Omega \times \Omega \rightarrow \mathbf{R}$ is continuous and, hence, the measure μ is σ -additive on Ω' . Therefore there is well defined the corresponding Gaussian random variable $\eta(\phi) \in \Omega'$.

In the case of the rigged Hilbert space (34) the Gaussian random field $\eta(\phi) \in \mathcal{S}'(\mathbf{R}^3)$ is nothing else than the *spatial white noise*. We extend this terminology and we shall call $\eta(\phi)$ *white noise* even in the abstract framework. Let us consider $\sqrt{\alpha}$ -scaling of white noise

$$\xi(\phi) = \sqrt{\alpha}\eta(\phi)$$

and its one-dimensional projections: $\xi_{\Psi}(\phi) = (\xi(\phi), \Psi), \Psi \in \Omega$. We have $E\xi_{\Psi} = 0, E\xi_{\Psi}^2 = \alpha\|\Psi\|^2$. The $\xi(\phi)$ is the *background field* in our prequantum model (PCSFT).

For any $\Psi \in \Omega$, we consider the one-dimensional projector $P_{\Psi}(\phi) = (\phi, \Psi)\Psi, \phi \in \Omega'$, and the Ω -valued random variable $P_{\Psi}\xi(\phi) = \xi_{\Psi}(\phi)\Psi$. If $\|\Psi\| = 1$, then the T -image of the corresponding Gaussian distribution ρ_{Ψ} is nothing else than the pure state Ψ .

This correspondence can be extended from the space Ω to the Hilbert space H . If $\Psi \in H$, then $\xi_\Psi(\phi) = (\phi, \Psi)$ is also well defined, but it is not a continuous linear functional on the space Ω' . The $\xi_\Psi(\phi)$ is defined as an element of the space of square integrable functionals of the white noise: $\xi_\Psi \in L_2(\Omega', d\mu)$. To define ξ_Ψ , we approximate $\Psi \in H$ by elements Ψ_n of Ω , $\Psi_n \rightarrow \Psi$ in H (we recall that Ω is dense in H). Then $\xi_\Psi = \lim_{n \rightarrow \infty} \xi_{\Psi_n}$ in $L_2(\Omega', d\mu)$.

Lemma 5.1. *Let $f : H \rightarrow \mathbf{R}$ be a polynomial and let $f(0) = 0$. Then, for any $\Psi \in H$, the asymptotic equality (29) holds.*

Proof. Here the main difference from consideration in section 3 is that the measure μ is not concentrated on Hilbert space H on which the function f is defined (and continuous). Therefore even the exponential growth of f on H would not help so much, because $\int_{\Omega'} e^{a\|\phi\|} d\mu(\phi) = \infty$ (since even $\int_{\Omega'} \|\phi\| d\mu(\phi) = \infty$). We have for a polynomial f :

$$\begin{aligned} Ef(P_\Psi \xi(\phi)) &= \sum_{k=1}^N \frac{f^{(2k)}(0)(\Psi, \dots, \Psi)}{2k!} E\xi_\Psi^{2k}(\phi) \\ &= \sum_{k=1}^N \frac{f^{(2k)}(0)(\Psi, \dots, \Psi)}{(2k)!} \frac{\alpha^{2k} \|\Psi\|^{2k} (2k)!}{2^k k!}. \end{aligned}$$

Since the sum is finite and derivatives of f are continuous forms on H , we obtain (29).

We “quantize” $f(u)$ by mapping it into $\frac{f''(0)}{2}$. For quadratic functionals $f(u) = \frac{1}{2}(Au, u)$, $A \in \mathcal{L}_s(H)$, we have the precise equality and we can directly use the average with respect to the canonical Gaussian random variable $\eta(\omega)$. Here

$$Ef(P_\Psi \eta(\omega)) = \frac{1}{2}(f''(0)\Psi, \Psi).$$

6 Unbounded operators

In this section we shall use theory of Gaussian measures on topological vector spaces, see, e.g., Smolyanov and Fomin [66] for detail.

Let $f : \Omega \rightarrow \mathbf{R}$ be a smooth function. Then, at any point $\psi_0 \in \Omega$, $f''(\psi_0) : \Omega \rightarrow \Omega'$. Therefore $f''(0)$ is in general unbounded operator in H .

Moreover, in this way (i.e., starting with PCSFT) we obtain the class of linear operators (quantum observables) that is even essentially

larger than in the conventional quantum formalism. In general, $A = f''(0)$ maps Ω not into H , but into Ω' .

Example 6.1. Let us consider the rigged Hilbert space (34). We consider the map $f : \mathcal{S}(\mathbf{R}^3) \rightarrow \mathbf{R}$ determined by a fixed point $x_0 \in \mathbf{R}^3$:

$$f(\psi) = \frac{1}{2}\psi^2(x_0).$$

(For example, the classical field $\psi(x) = e^{-x^2}$ is mapped into the real number $e^{-x_0^2}$). Then $(f''(0)\psi_1, \psi_2) = \psi_1(x_0)\psi_2(x_0)$. Thus

$$A\psi(x) = \frac{1}{2}f''(0)\psi(x) = \frac{1}{2}\delta(x - x_0)\psi(x)$$

is the operator of multiplication by the δ -function $\delta(x - x_0)$. Hence

$$f''(0)(\mathcal{S}(\mathbf{R}^3)) \not\subset L_2(\mathbf{R}^3).$$

For any $\Psi \in \mathcal{S}(\mathbf{R}^3)$, we have

$$Ef(P_\Psi\eta(\omega)) = \frac{1}{2}\Psi^2(x_0) = (A\Psi, \Psi) \equiv \langle A \rangle_\Psi.$$

However, in general for $\Psi \in L_2(\mathbf{R}^3)$ the average $\langle A \rangle_\Psi$ is not well defined.

We can consider not only pure states, but general density operators. Let us now consider a Gaussian measure $\rho \in S_G^\alpha(H)$ which has the support on the space Ω . Thus ρ can be considered as a measure on Ω . For such a measure ρ its covariation operator $B : \Omega' \rightarrow \Omega$ and its Fourier transform $\tilde{\rho}$ is defined on Ω' . We denote this class of statistical states by the symbol $S_G^\alpha(\Omega)$. We remark that $S_G^\alpha(\Omega) \subset S_G^\alpha(H)$.

Let E be a complex locally convex topological linear space. We recall that the topology of E can be determined by a system of semi-norms (the notion of a semi-norm p generalizes the notion of a norm $\|\cdot\|$; the only difference is that $p(\psi)$ can be equal to zero even for a nonzero vector ψ). Let $b_n : E \times \dots \times E \rightarrow \mathbf{C}$ be a continuous n -linear symmetric form. There exists a continuous semi-norm p on E such that

$$\|b_n\|_p = \sup_{p(\psi) \leq 1} |b_n(\psi, \dots, \psi)| < \infty$$

(here $p \equiv p_{b_n}$). Thus

$$|b_n(\psi, \dots, \psi)| \leq \|b_n\| p^n(\psi) \quad (35)$$

An analytic function, see, e.g., [60] for details, $f : E \rightarrow \mathbf{C}$ has the exponential growth if there exists a continuous semi-norm p on E on such that:

$$|f(\psi)| \leq a e^{bp(\psi)}, \psi \in E. \quad (36)$$

Here the constants and the semi-norm depend on f : $a \equiv a_f, b \equiv b_f, p \equiv p_f$.

Lemma 6.1. *The space of analytic functions of the exponential growth coincides with the space of analytic functions such that there exists a continuous semi-norm $p = p_f$:*

$$\|f^{(n)}(0)\|_p \leq c r^n, \quad n = 0, 1, 2, \dots \quad (37)$$

Here constants $c = c_f$ and $r = r_f$ depend on the function f .

Proof. A). Let f have the exponential growth. For any $\psi \in E$, we consider the function of the complex variable $z \in \mathbf{C}$: $g_\psi(z) = f(z\psi)$. As in Lemma 5.1, we have: $|g_\psi^{(n)}(0)| \leq n! R^{-n} \sup_{0 \leq \theta \leq 2\pi} |f(Re^{i\theta}\psi)| \leq a_f n! R^{-n} e^{b_f R p(\psi)}$. We obtain:

$$\|f^{(n)}(0)\|_p \leq a'_f e^{-n} n^{1/2} e^{b_f n}.$$

Thus the derivatives of f satisfy the inequalities (37) with $r_f = e^{b_f}$.

B). Let now derivatives of f satisfy the inequalities (37) for some continuous semi-norm p . Then by the inequalities (35) we have

$$|f(\psi)| \leq \sum_{n=0}^{\infty} \|f^{(n)}(0)\|_p p^n(\psi)/n! \leq c_f e^{r_f p(\psi)}.$$

Thus f has the exponential growth with $b_f = r_f$ and the same continuous semi-norm p as in (37).

We denote by $\Omega^{\mathbf{C}}$ the complexification of Ω : $\Omega^{\mathbf{C}} = \Omega \oplus i\Omega$. We denote by $\mathcal{V}(\Omega)$ the class of functions $f : \Omega \rightarrow \mathbf{R}, f(0) = 0$, which can be analytically continued onto $\Omega^{\mathbf{C}}$ and they have the exponential growth.

Lemma 6.2. *Let $\rho \in S_G^\alpha(\Omega)$. Then, for any function $f \in \mathcal{V}(\Omega)$, the following asymptotic equality holds:*

$$\langle f \rangle_\rho \equiv \int_\Omega f(\psi) d\rho(\psi) = \frac{\alpha}{2} \int_\Omega (f''(0)\psi, \psi) d\rho_D(u) + o(\alpha), \quad \alpha \rightarrow 0, \quad (38)$$

where $D = \frac{\text{cov}\rho}{\alpha}$. Here

$$o(\alpha) = \alpha^2 R(\alpha, f, \rho), \quad (39)$$

where

$$|R(\alpha, f, \rho)| \leq c_f \int_{\Omega} e^{r_f p(\psi)} d\rho_D(\psi). \quad (40)$$

The semi-norm p is determined by the inequality (36).

The proof of this Theorem repeats the proof of Lemma 3.2. Instead of Lemma 3.1, we apply its generalization to the case of an arbitrary locally convex topological linear space, see Lemma 6.1.

We pay attention that $D : \Omega' \rightarrow \Omega$, and $A = \frac{f''(0)}{2} : \Omega \rightarrow \Omega'$, so $C = DA : \Omega \rightarrow \Omega$. In general, this operator can not be extended to a continuous operator in H . We would like to obtain an analogue of the formula (8) for linear continuous operators $A : \Omega \rightarrow \Omega'$:

$$\int_{\Omega} (A\psi, \psi) d\rho_D(u) = \text{Tr } DA \quad (41)$$

The main mathematical problem is that in general the operator $C = DA$ is not even continuous in H , so it is not a trace class operator in the Hilbert space H . Nevertheless, we can introduce the notion of trace even in such a framework.

We recall that systems of vectors $\{e_j\}_{j=1}^{\infty}, e_j \in \Omega$, and $\{e'_j\}_{j=1}^{\infty}, e'_j \in \Omega'$, are called *biorthogonal topological bases* in Ω and Ω' if

$$(e'_j, e_i) = \delta_{ij}, \text{ and } \psi = \sum_{j=1}^{\infty} (e'_j, \psi) e_j, \psi \in \Omega, \phi = \sum_{j=1}^{\infty} (\phi, e_j) e'_j, \phi \in \Omega',$$

where the series converge in Ω and Ω' , respectively.

Definition 6.1. A linear continuous operator $C : \Omega \rightarrow \Omega$ is called *trace-class operator* if, for any pair of biorthogonal topological bases, the series

$$\text{Tr } C = \sum_{j=1}^{\infty} (e'_j, C e_j)$$

converges and its sum does not depend on bases.

Lemma 6.3. Let ρ be a Gaussian measure on Ω and let $A : \Omega \rightarrow \Omega'$ be a continuous operator. Then the operator $C = DA$, where $D = \text{cov}\rho$, belongs to the trace class and the equality (41) holds.

As a consequence of Lemmas 6.2 and 6.3, we obtain:

Theorem 6.1. *Let $\rho \in S_G^\alpha(\Omega)$. Then, for any function $f \in \mathcal{V}(\Omega)$, the following asymptotic equality holds:*

$$\langle f \rangle_\rho \equiv \int_\Omega f(\psi) d\rho(\psi) = \text{Tr } Df''(0)/2 + o(\alpha), \quad \alpha \rightarrow 0, \quad (42)$$

where $D = \frac{\text{cov} \rho}{\alpha}$.

Thus our prequantum model, PCSFT, provides the motivation to extend the set of quantum observables and consider all continuous operators $A : \Omega \rightarrow \Omega'$. Operators should be self-adjoint in the ordinary sense: $A' = A$. We recall that here $A' : \Omega'' \rightarrow \Omega'$, but $\Omega'' \equiv \Omega$, since Ω is a nuclear Frechet space and hence it is reflexive. Denote the set of such operators by the symbol $\mathcal{L}_s(\Omega, \Omega')$. Denote the set of covariation operators of Gaussian measures belonging the space $S_G^1(\Omega)$ by the symbol $\mathcal{D}(\Omega', \Omega)$.

Definition 6.2. *A statistical quantum model corresponding to a rigged Hilbert space \mathcal{T} given by (33) is the pair*

$$N_{\text{quant}}(\mathcal{T}) = (\mathcal{D}(\Omega', \Omega), \mathcal{L}_s(\Omega, \Omega')).$$

A generalized density operators $D \in \mathcal{D}(\Omega', \Omega)$ represents a statistical state; a linear operator $A \in \mathcal{L}_s(\Omega, \Omega')$ represents a quantum observable. The average of such an observable with respect to such a statistical state is given by the following generalization of the von Neumann trace-formula:

$$\langle A \rangle_D = \text{Tr } DA \quad (43)$$

We choose the state space Ω – a nuclear Frechet space. For a rigged Hilbert space \mathcal{T} given by (33), we consider the classical statistical model $M^\alpha(\mathcal{T}) = (S_G^\alpha(\Omega), \mathcal{V}(\Omega))$. Here as always $\langle f \rangle_\rho = \int_\Omega f(\psi) d\rho(\psi)$.

The equality (42) can be used as the motivation for defining the following classical \rightarrow quantum map T from the classical statistical model $M^\alpha(\mathcal{T}) = (S_G^\alpha(\Omega), \mathcal{V}(\Omega))$ onto the quantum statistical model $N_{\text{quant}}(\mathcal{T}) = (\mathcal{D}(\Omega', \Omega), \mathcal{L}_s(\Omega, \Omega'))$ by (27), (28). Our previous considerations can be presented as

Theorem 6.2. *The map $T : S_G^\alpha(\Omega) \rightarrow \mathcal{D}(\Omega', \Omega)$ is one-to-one; the map $T : \mathcal{V}(\Omega) \rightarrow \mathcal{L}_s(\Omega, \Omega')$ is linear surjection and the classical and quantum averages are coupled by the asymptotic equality (42).*

Example 6.2. The position operators $\hat{x}_j, j = 1, 2, 3$ can be obtained as $\hat{x}_j = \frac{1}{2}f''_{x_j}(0)$, where

$$f_{x_j}(\psi) = \int_{\mathbf{R}^3} x_j \psi^2(x) dx.$$

Here the operator of multiplication $\hat{x}_j : \mathcal{S}(\mathbf{R}^3) \rightarrow \mathcal{S}(\mathbf{R}^3), \psi \rightarrow x_j \psi$, is continuous. Hence $\hat{x}_j : \mathcal{S}(\mathbf{R}^3) \rightarrow \mathcal{S}'(\mathbf{R}^3)$ is also continuous. Thus, for any measure $\rho \in S_G^\alpha(\mathcal{S}(\mathbf{R}^3))$, we have

$$\langle f_{x_j} \rangle_\rho \equiv \int_{\mathcal{S}(\mathbf{R}^3)} \int_{\mathbf{R}^3} x \psi^2(x) dx d\rho(\psi) = \alpha \text{Tr } D \hat{x}_j,$$

$D = \text{cov } \rho / \alpha$ (here the trace of the composition $D \hat{x}_j$ is well defined).

Example 6.3. Let x_0 be a fixed point in \mathbf{R}^3 . Let now $A\psi(x) = \delta(x - x_0)\psi(x), \psi \in \mathcal{S}(\mathbf{R}^3)$. This operator does not belong to the domain of the conventional quantum formalism. It could not be represented as an unbounded operator in $H = L_2(\mathbf{R}^3)$ with a dense domain of definition. Nevertheless,

$$\int_{\mathcal{S}(\mathbf{R}^3)} (A\psi, \psi) d\rho(\psi) = \int_{\mathcal{S}(\mathbf{R}^3)} \psi^2(x_0) d\rho(\psi) = \alpha \text{Tr } DA$$

and the trace of the composition DA is well defined.

Example 6.4. The momentum operators $\hat{p}_j, j = 1, 2, 3$, can be obtained as $\hat{p}_j = \frac{1}{2}f''_{p_j}(0)$, where

$$f_{p_j}(\psi) = -i \int_{\mathbf{R}^3} \frac{\partial \psi}{\partial x_j}(x) \overline{\psi(x)} dx.$$

Here the operator $\hat{p}_j : \mathcal{S}(\mathbf{R}^3) \rightarrow \mathcal{S}(\mathbf{R}^3)$ is continuous. Hence, $\hat{p}_j : \mathcal{S}(\mathbf{R}^3) \rightarrow \mathcal{S}'(\mathbf{R}^3)$ is also continuous. Thus for any measure $\rho \in S_{G,\text{symp}}^\alpha(\mathcal{S}(\mathbf{R}^3) \times \mathcal{S}(\mathbf{R}^3))$, we have (for $\psi(x) = q(x) + ip(x)$):

$$\langle f_{p_j} \rangle_\rho \equiv -i \int_{\mathcal{S}(\mathbf{R}^3) \times \mathcal{S}(\mathbf{R}^3)} \int \frac{\partial \psi}{\partial x_j}(x) \overline{\psi(x)} dx d\rho(\psi) = \alpha \text{Tr } D \hat{p}_j,$$

where $D = \text{cov } \rho / \alpha$. Here $D \hat{p}_j : \mathcal{S}(\mathbf{R}^3) \rightarrow \mathcal{S}(\mathbf{R}^3)$ is the trace class operator. Similar considerations can be done for angular momentum operators.

7 Generalized quantum mechanics: approximations of higher orders

We have created the classical statistical model which induced the quantum statistical model. The quantum description can be obtained through the Taylor expansion of classical physical variables up to the terms of the second order. The crucial point is the presence of a parameter α which small in QM, but not in the prequantum classical model.

This viewpoint to conventional quantum mechanics implies the evident possibility to generalize this formalism by considering higher orders of the Taylor expansion of classical physical variables and corresponding expansions of classical averages with respect to the parameter α .

We still consider the real case: $\Omega = H$, where H is the real separable Hilbert space, and only bounded linear operators (and forms). We recall that momentums of a measure ρ are defined by

$$a_\rho^{(k)}(z_1, \dots, z_k) = \int_\Omega (z_1, \psi) \dots (z_k, \psi) d\rho(\psi).$$

In particular, $a_\rho^{(1)} \equiv a_\rho$ is the mean value and $a_\rho^{(2)}$ is the covariation form. We remark that for a Gaussian measure ρ , $a_\rho = 0$ implies that all its momenta of odd orders $a_\rho^{(k)}$, $k = 2n + 1$, $n = 0, 1, \dots$, are also equal to zero.

Therefore the expansion of $\langle f \rangle_\rho$ with respect to $s = \alpha^{1/2}$ does not contain terms with s^{2n+1} . Hence this is the expansion with respect to $\alpha^n (= s^{2n})$, $n = 1, 2, \dots$. We are able to create $o(\alpha^n)$ -generalization of quantum mechanics through neglecting by terms of the magnitude $o(\alpha^n)$, $\alpha \rightarrow 0$ ($n = 1, 2, \dots$) in the power expansion of the classical average. Of course, for $n = 1$ we obtain the conventional quantum mechanics. Let us consider the classical statistical model

$$M^\alpha = (S_G^\alpha(\Omega), \mathcal{V}(\Omega)). \quad (44)$$

By taking into account that $a_\rho^{2n+1} = 0$, $n = 0, 1, \dots$, for $\rho \in S_G^\alpha(\Omega)$, we have:

$$\langle f \rangle_\rho = \frac{\alpha}{2} \text{Tr } Df''(0) + \sum_{k=2}^{\infty} \frac{\alpha^k}{(2k)!} \int_\Omega f^{(2k)}(0)(\phi, \dots, \phi) d\rho_D(\phi), \quad (45)$$

where as always $D = \frac{\text{cov}\rho}{\alpha}$.

We now consider a new epistemic (“observational”) statistical model which is a natural generalization of the conventional quantum mechanics. We start with some preliminary mathematical considerations. Let A and B be two n -linear symmetric forms. We define their trace by

$$\text{Tr } BA = \sum_{j_1, \dots, j_n=1}^{\infty} B(e_{j_1}, \dots, e_{j_n}) A(e_{j_1}, \dots, e_{j_n}), \quad (46)$$

if this series converges and its sum does not depend on the choice of an orthonormal basis $\{e_j\}$ in Ω . We remark that

$$\langle f \rangle_{\rho} = \frac{\alpha}{2} \text{Tr } Df''(0) + \sum_{k=2}^n \frac{\alpha^k}{2k!} \text{Tr } a_{\rho_D}^{(2k)} f^{(2k)}(0) + o(\alpha^n), \alpha \rightarrow 0, \quad (47)$$

Here we used the following result about Gaussian integrals:

Lemma 7.1. *Let A_k be a continuous k -linear form on Ω and let ρ_D be a Gaussian measure (with zero mean value and the covariation operator D). Then*

$$\int_{\Omega} A_k(\psi, \dots, \psi) d\rho_D(\psi) = \text{Tr } a_{\rho_D}^{(k)} A_k. \quad (48)$$

Proof. Let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal basis in Ω . We apply the well known Lebesgue theorem on majorant convergence. We set

$$f_N(\psi) = \sum_{j_1, \dots, j_k=1}^n A_k(e_{j_1}, \dots, e_{j_k})(e_{j_1}, \psi) \dots (e_{j_k}, \psi). \quad (49)$$

We have

$$|f_N(\psi)| = |A_k(\sum_{j_1=1}^N (x, e_{j_1}) e_{j_1} \dots, \sum_{j_k=1}^N (\psi, e_{j_k}) e_{j_k})| \leq \|A_k\| \|\psi\|^k. \quad (50)$$

Therefore we obtain:

$$\begin{aligned} \int_{\Omega} A_k(\psi, \dots, \psi) d\rho_D(\psi) &= \lim_{N \rightarrow \infty} \int_{\Omega} f_N(\psi) d\rho_D(\psi) \\ &= \sum_{j_1=1, \dots, j_k=1}^{\infty} A_k(e_{j_1}, \dots, e_{j_k}) \int_{\Omega} (e_{j_1}, \psi) \dots (e_{j_k}, \psi) d\rho_D(\psi) = \text{Tr } a_{\rho_D}^{(k)} A_k. \end{aligned} \quad (51)$$

The proof is finished.

In particular, we obtained the following inequality:

$$|\text{Tr } a_{\rho_D}^k A_k| \leq \|A\| \int_{\Omega} \|\psi\|^k d\rho_D(\psi). \quad (52)$$

We now remark that for a Gaussian measure (with zero mean value) integrals (48) are equal to zero for $k = 2l+1$. Thus $\text{Tr } a_{\rho_D}^{(2l+1)} A_{2l+1} = 0$. It is easy to see that $2k$ -linear forms (momenta of even order) $a_{\rho_D}^{2k}$ can be expressed through the covariance operator D :

$$a_{\rho_D}^{(2k)} = e(k, D) = \frac{d^{2k}}{d\phi^{2k}} e^{-\frac{1}{2}(D\phi, \phi)}|_{\phi=0}. \quad (53)$$

In particular, $e(2, D)(\phi_1, \phi_2) = (D\phi_1, \phi_2)$ and $e(4, D)(\phi_1, \phi_2, \phi_3, \phi_4)$

$$= (D\phi_1, \phi_3)(D\phi_2, \phi_4) + (D\phi_2, \phi_3)(D\phi_1, \phi_4) + (D\phi_1, \phi_2)(D\phi_3, \phi_4). \quad (54)$$

Thus (47) can be rewritten as

$$\langle f \rangle_{\rho_B} = \frac{\alpha}{2} \text{Tr } Df''(0) + \sum_{k=2}^n \frac{\alpha^k}{2k!} \text{Tr } e(2k, D) f^{(2k)}(0) + o(\alpha^n), \quad \alpha \rightarrow 0, \quad (55)$$

or by introducing the $1/\alpha$ -amplification of the classical physical variable f we have:

$$\langle f_{\alpha} \rangle_{\rho_B} = \frac{1}{2} \text{Tr } Df''(0) + \sum_{k=2}^n \frac{\alpha^{k-1}}{2k!} \text{Tr } e(2k, D) f^{(2k)}(0) + o(\alpha^{n-1}) \quad (56)$$

This formula is the basis of *a new quantum theory*. In this theory statistical states can be still represented by von Neumann density operators $D \in \mathcal{D}(\Omega)$, but observables are represented by multiples $A = (A_2, A_4, \dots, A_{2n})$, where A_{2j} are symmetric $2n$ -linear forms on a Hilbert space Ω . In particular, the quadratic form A_2 can be represented by a self-adjoint operator. To escape mathematical difficulties, we can assume that forms A_{2j} are continuous. Denote the space of all such multiples A by $L_{2n}(\Omega)$. We obtain the following generalization of the conventional quantum model:

$$N_{\text{quant}, 2n} = (\mathcal{D}(\Omega), L_{2n}(\Omega)). \quad (57)$$

Here the average of an observable $A \in L_{2n}(\Omega)$ with respect to a state $D \in \mathcal{D}(\Omega)$ is given by

$$\langle a \rangle_D = \sum_{n=1}^n \text{Tr } e(2k, D) A_{2k} \quad (58)$$

If one define $\text{Tr } DA = \sum_{k=1}^n \text{Tr } e(2k, D) A_{2k}$ then the formula (58) can be written as in the conventional quantum mechanics (von Neumann's formula of n th order):

$$\langle A \rangle_D = \text{Tr } DA \quad (59)$$

This model is the result of the following “quantization” procedure of the classical statistical model $M^\alpha = (S_G^\alpha(\Omega), \mathcal{V}(\Omega))$:

$$\rho \rightarrow D = \frac{\text{cov} \rho}{\alpha}; \quad (60)$$

$$f \rightarrow A = \left(\frac{1}{2} f''(0), \frac{\alpha}{4!} f^{(4)}(0), \dots, \frac{\alpha^{n-1}}{(2n)!} f^{(2n)}(0) \right). \quad (61)$$

(thus here $A_{2k} = \frac{\alpha^{k-1}}{(2k)!} f^{(2k)}(0)$). The transformation T_{2n} given by (60), (61) maps the classical statistical model $M^\alpha = (S_G^\alpha(\Omega), \mathcal{V}(\Omega))$ onto generalized quantum model $N_{\text{quant}, 2n} = (\mathcal{D}(\Omega), L_{2n}(\Omega))$.

Theorem 7.1. *For the classical statistical model $M^\alpha = (S_G^\alpha(\Omega), \mathcal{V}(\Omega))$, the classical \rightarrow quantum map T_{2n} , defined by (60) and (61), is one-to-one for statistical states; it has a huge degeneration for variables. Classical and quantum averages are coupled through the asymptotic equality (55).*

We pay attention to the simple mathematical fact that the degree of degeneration of the map $T_{2n} : \mathcal{V}(\Omega) \rightarrow L_{2n}(\Omega)$ is decreasing for $n \rightarrow \infty$. Denote the space of polynomials of the degree $2n$ containing only terms of even degrees by the symbol P_{2n} . Thus $f \in P_{2n}$ iff $f(\psi) = Q_2(\psi, \psi) + Q_4(\psi, \psi, \psi, \psi) + \dots + Q_{2n}(\psi, \dots, \psi)$, where $Q_{2j} : \Omega^{2j} \rightarrow \mathbf{R}$ is a symmetric $2j$ -linear (continuous) form. The restriction of the map T_{2n} on the subspace P_{2n} of the space \mathcal{V} is one-to-one. One can also consider a generalized quantum model

$$N_{\text{quant}, \infty} = (\mathcal{D}, L_\infty), \quad (62)$$

where $L_\infty(\Omega)$ consists of infinite sequences of $2n$ -linear (continuous) forms on Ω :

$$A = (A_2, \dots, A_{2n}, \dots). \quad (63)$$

The correspondence between the classical model M^α (for any α) and the generalized quantum model $N_{\text{quant},\infty}$ is one-to-one.

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References

- [1] D. Hilbert, J. von Neumann, L. Nordheim, *Math. Ann.*, **98**, 1-30 (1927).
- [2] P. A. M. Dirac, *The Principles of Quantum Mechanics*, Oxford Univ. Press, 1930.
- [3] W. Heisenberg, *Physical principles of quantum theory*, Chicago Univ. Press, 1930.
- [4] J. von Neumann, *Mathematical foundations of quantum mechanics*, Princeton Univ. Press, Princeton, N.J., 1955.
- [5] A. Einstein, B. Podolsky, N. Rosen, *Phys. Rev.* **47**, 777–780 (1935).
- [6] E. Schrödinger, *Philosophy and the Birth of Quantum Mechanics*. Edited by M. Bitbol, O. Darrigol (Editions Frontières, Gif-sur-Yvette, 1992); especially the paper of S. D’Agostino, “Continuity and completeness in physical theory: Schrödinger’s return to the wave interpretation of quantum mechanics in the 1950’s”, pp. 339-360.
- [7] E. Schrödinger, *E. Schrödinger Gesammelte Abhandlungen* (Wieweg and Son, Wien, 1984); especially the paper “What is an elementary particle?”, pp. 456-463.
- [8] A. Einstein, *The collected papers of Albert Einstein* (Princeton Univ. Press, Princeton, 1993).
- [9] A. Einstein and L. Infeld, *The evolution of Physics. From early concepts to relativity and quanta* (Free Press, London, 1967).
- [10] A. Lande, *New foundations of quantum mechanics*, Cambridge Univ. Press, Cambridge, 1965.
- [11] L. De Broglie, *The current interpretation of wave mechanics, critical study*. Elsevier Publ., Amsterdam-London-New York, 1964.
- [12] J. S. Bell, *Speakable and unspeakable in quantum mechanics*, Cambridge Univ. Press, 1987.
- [13] G. W. Mackey, *Mathematical foundations of quantum mechanics*, W. A. Benjamin INC, New York, 1963.
- [14] S. Kochen and E. Specker, *J. Math. Mech.*, **17**, 59-87 (1967).
- [15] L. E. Ballentine, *Rev. Mod. Phys.*, **42**, 358–381 (1970).
- [16] G. Ludwig, *Foundations of quantum mechanics*, Springer, Berlin, 1983.

- [17] E. B. Davies, J. T. Lewis, *Comm. Math. Phys.* **17**, 239-260 (1970).
- [18] E. Nelson, *Quantum fluctuation*, Princeton Univ. Press, Princeton, 1985.
- G.C. Ghirardi, C. Omero, A. Rimini and T. Weber, The Stochastic Interpretation of Quantum Mechanics: a Critical Review, *Rivista del Nuovo Cimento* **1** 1 (1978).
- S. Alberverio, and R. Höegh-Krohn, A remark on the connection between stochastic mechanics and the heat equation. *J. Math. Phys.*, **15**, 1745-1747 (1975).
- J. Lörinczi, R. A. Minlos, and H. Spohn, The infrared behaviour in Nelson's model of a quantum particle coupled to a massive scalar field. *Ann. H. Poincare*, **3**, 269-295 (2002).
- [19] D. Bohm and B. Hiley, *The undivided universe: an ontological interpretation of quantum mechanics*, Routledge and Kegan Paul, London, 1993.
- [20] S. P. Gudder, *Trans. AMS* **119**, 428-442 (1965).
- [21] S. P. Gudder, *Axiomatic quantum mechanics and generalized probability theory*, Academic Press, New York, 1970.
- [22] H. Spohn, Quantum measurement theory including initial correlations and observables with continuous spectrum. *Int. J. Theor. Phys.*, **15**, 283-375 (1976).
- [23] R. Feynman and A. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw-Hill, New-York, 1965.
- [24] J. M. Jauch, *Foundations of Quantum Mechanics*, Addison-Wesley, Reading, Mass., 1968.
- [25] A. Peres, *Quantum Theory: Concepts and Methods*, Dordrecht, Kluwer Academic, 1994.
- [26] L. Accardi, "The probabilistic roots of the quantum mechanical paradoxes" in *The wave-particle dualism. A tribute to Louis de Broglie on his 90th Birthday*, edited by S. Diner, D. Fargue, G. Lochak and F. Selleri, D. Reidel Publ. Company, Dordrecht, 1984, pp. 297-330.
- [27] L. Accardi, *Urne e Camaleoni: Dialogo sulla realta, le leggi del caso e la teoria quantistica*, Il Saggiatore, Rome, 1997.
- [28] L. E. Ballentine, *Quantum mechanics*, Englewood Cliffs, New Jersey, 1989.
- [29] L. E. Ballentine, "Interpretations of probability and quantum theory", in *Foundations of Probability and Physics*, edited by A. Yu. Khrennikov, Q. Prob. White Noise Anal., 13, WSP, Singapore, 2001, pp. 71-84.

- [30] A. S. Holevo, *Probabilistic and statistical aspects of quantum theory*, North-Holland, Amsterdam, 1982.
- [31] A. S. Holevo, *Statistical structure of quantum theory*, Springer, Berlin-Heidelberg, 2001.
- [32] P. Busch, M. Grabowski, P. Lahti, *Operational Quantum Physics*, Springer Verlag, Berlin, 1995.
- [33] A. Yu. Khrennikov (editor), *Foundations of Probability and Physics*, Q. Prob. White Noise Anal., 13, WSP, Singapore, 2001.
- [34] A. Yu. Khrennikov (editor), *Quantum Theory: Reconsideration of Foundations*, Ser. Math. Modeling, 2, Växjö Univ. Press, 2002.
- [35] A. Yu. Khrennikov (editor), *Foundations of Probability and Physics-2*, Ser. Math. Modeling, 5, Växjö Univ. Press, 2003.
- [36] A. Yu. Khrennikov (editor), *Quantum Theory: Reconsideration of Foundations-2*, Ser. Math. Modeling, 10, Växjö Univ. Press, 2004.
- [37] A. Yu. Khrennikov (editor), Proceedings of Conference *Foundations of Probability and Physics-3*, American Institute of Physics, Ser. Conference Proceedings, **750**, 2005.
- [38] A. Yu. Khrennikov, *Interpretations of Probability*, VSP Int. Sc. Publishers, Utrecht/Tokyo, 1999 (second edition, 2004).
- [39] A. E. Allahverdyan, R. Balian, T. M. Nieuwenhuizen, in: A. Yu. Khrennikov (Ed.), *Foundations of Probability and Physics-3*, Melville, New York: AIP Conference Proceedings, 2005, pp. 16-24.
- [40] W. De Baere, *Lett. Nuovo Cimento* **39**, 234 (1984); **40**, 488 (1984); *Advances in electronics and electron physics* **68**, 245 (1986).
- [41] De Muynck W. M., *Foundations of Quantum Mechanics, an Empiricists Approach* (Kluwer, Dordrecht) 2002.
- [42] De Muynck W., De Baere W., Martens H., *Found. of Physics* **24** (1994) 1589.
- [43] Allahverdyan A. E., Balian R., Nieuwenhuizen Th., *Europhys. Lett.* **61** (2003) 452.
- [44] K. Hess and W. Philipp, *Proc. Nat. Acad. Sc.* **98**, 14224 (2001); **98**, 14227 (2001); **101**, 1799 (2004); *Europhys. Lett.* **57**, 775 (2002).
- [45] A. Yu. Khrennikov, *J. Phys. A: Math. Gen.* **34**, 9965-9981 (2001); *Il Nuovo Cimento B* **117**, 267-281 (2002); *J. Math. Phys.* **43**, 789-802 (2002); *Information dynamics in cognitive, psychological and anomalous phenomena*, Ser. Fundamental Theories of Physics, Kluwer, Dordrecht, 2004; *J. Math. Phys.* **44**, 2471- 2478 (2003); *Phys.*

Lett. A **316**, 279-296 (2003); *Annalen der Physik* **12**, 575-585 (2003).

[46] A. Yu. Khrennikov, A pre-quantum classical statistical model with infinite-dimensional phase space. *J. Phys. A: Math. Gen.*, **38**, 9051-9073 (2005).

[47] A. Yu. Khrennikov, Quantum mechanics as an asymptotic projection of statistical mechanics of classical fields: derivation of Schrödinger's, Heisenberg's and von Neumann's equations. <http://www.arxiv.org/abs/quant-ph/051>

[48] O. G. Smolyanov, Infinite-dimensional pseudodifferential operators and Schrödinger quantization. *Dokl. Akad. Nauk USSR*, **263**, 558(1982).

[49] A. Yu. Khrennikov, Infinite-dimensional pseudo-differential operators. *Izvestia Akademii Nauk USSR, ser.Math.*, **51**, 46 (1987).

[50] L. de la Pena and A. M. Cetto, *The Quantum Dice: An Introduction to Stochastic Electrodynamics* Kluwer. Dordrecht, 1996; T. H. Boyer, *A Brief Survey of Stochastic Electrodynamics* in Foundations of Radiation Theory and Quantum Electrodynamics, edited by A. O. Barut, Plenum, New York, 1980; T. H. Boyer, Timothy H., *Scientific American*, pp 70-78, Aug 1985; see also an extended discussion on vacuum fluctuations in: M. O. Scully, M. S. Zubairy, *Quantum optics*, Cambridge University Press, Cambridge, 1997; W. H. Louisell, *Quantum Statistical Properties of Radiation*. J. Wiley, New York, 1973; L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics*. Cambridge University Press, Cambridge, 1995.

[51] L. De La Pena, *Found. Phys.* **12**, 1017 (1982); *J. Math. Phys.* **10**, 1620 (1969); L. De La Pena, A. M. Cetto, *Phys. Rev. D* **3**, 795 (1971).

[52] A. Bach, *J. Math. Phys.* **14** 125 (1981).

[53] A. Bach, *Phys. Lett. A* **73** 287 (1979).

[54] A. Bach, *J. Math. Phys.* **21** 789 (1980).

[55] A. V. Skorohod, *Integration in Hilbert space*. Springer-Verlag, Berlin, 1974.

[56] T. Hida, *Selected Papers of Takeyiki Hida*. H. H. Kuo, N. Obata, K. Saito, L. Streit, Si Si, L. Accardi (editors) World Scientific Publ. (2001).

[57] T. Hida, M. Hitsuda *Gaussian Processes*, Translations of Mathematical Monographs, **120**, American Mathematical Society, 1993.

[58] S. Albeverio, and M. Röckner, *Prob. Theory and Related Fields* **89**, 347 (1991).

S. Albeverio, R. Höegh-Krohn, Dirichlet forms and diffusion processes on rigged Hilbert spaces. *Zeitschrift für Wahrscheinlichkeitsthe-*

orie und verwandte Gebite, **40**, 59-106 (1977).

[59] B. Simon, *Functional Integration and Quantum Physics*. Ams Chelsea Pub., 2005)

[60] A. Yu. Khrennikov, Equations with infinite-dimensional pseudo-differential operators. Dissertation for the degree of candidate of phys-math. sc., Dept. Mechanics-Mathematics, Moscow State University, Moscow, 1983.

[61] O. G. Smolyanov and S. V. Fomin, Measures on topological linear spaces. *Russian Math. Surveys*, **31**, 3-5 (1976).